Determinants

Cofactors and 2×2 Determinants

Definition 3.1: Determinant of a Two By Two Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then*

$$\det(A) = ad - cb$$

let A be a square matrix. The **determinant** of A, denoted by $\det(A)$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 3.2: A Two by Two Determinant

Find $\det(A)$ *for the matrix* $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$.

$$\det(A) = (2)(6) - (-1)(4) = 12 + 4 = 16$$

Definition 3.3: The ij^{th} Minor of a Matrix

Let A *be a* 3×3 *matrix. The* ij^{th} ***minor*** *of* A, *denoted as* $minor(A)_{ij}$, *is the determinant of the* 2×2 *matrix which results from deleting the* i^{th} *row and the* j^{th} *column of* A.
In general, if A *is an* $n \times n$ *matrix, then the* ij^{th} *minor of* A *is the determinant of the* $n-1 \times n-1$ *matrix which results from deleting the* i^{th} *row and the* j^{th} *column of* A.

Example 3.4: Finding Minors of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $minor(A)_{12}$ and $minor(A)_{23}$.

$$minor(A)_{12} = \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

$$minor(A)_{23} = \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Definition 3.5: The ij^{th} Cofactor of a Matrix

*Suppose A is an $n \times n$ matrix. The ij^{th} **cofactor**, denoted by $cof(A)_{ij}$ is defined to be*

$$cof(A)_{ij} = (-1)^{i+j}\, minor\,(A)_{ij}$$

Example 3.6: Finding Cofactors of a Matrix

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $cof(A)_{12}$ and $cof(A)_{23}$.

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = -2$$

$$\text{cof}\,(A)_{12} = (-1)^{1+2}\, minor\,(A)_{12} = (-1)^{1+2}\,(-2) = 2$$

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

$$\text{cof}\,(A)_{23} = (-1)^{2+3}\, minor\,(A)_{23} = (-1)^{2+3}\,(-4) = 4$$

Definition 3.7: The Determinant of a Three By Three Matrix

*Let A be a 3×3 matrix. Then, $\det(A)$ is calculated by picking a row (or column) and taking the product of each entry in that row (column) with its cofactor and adding these products together. This process when applied to the i^{th} row (column) is known as **expanding along the i^{th} row (column)** as is given by*

$$\det(A) = a_{i1}\text{cof}(A)_{i1} + a_{i2}\text{cof}(A)_{i2} + a_{i3}\text{cof}(A)_{i3}$$

Example 3.8: Finding the Determinant of a Three by Three Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$ using the method of Laplace Expansion.

$$\det(A) = 1\overbrace{(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{11}} + 4\overbrace{(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{21}} + 3\overbrace{(-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}}^{\text{cof}(A)_{31}}$$

$$\det(A) = 1(1)(-1) + 4(-1)(-4) + 3(1)(-5) = -1 + 16 + -15 = 0$$

$$\det(A) = \overbrace{4(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{21}} + \overbrace{3(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}}^{\text{cof}(A)_{22}} + \overbrace{2(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}}^{\text{cof}(A)_{23}}$$

$$\det(A) = 4(-1)(-2) + 3(1)(-8) + 2(-1)(-4) = 0$$

Example 3.10: Determinant of a Four by Four Matrix

Find $\det(A)$ *where*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{bmatrix}$$

$$\det(A) = 3(-1)^{1+3}\begin{vmatrix} 5 & 4 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} +$$

$$4(-1)^{3+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 3 & 4 & 2 \end{vmatrix} + 3(-1)^{4+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 3 & 5 \end{vmatrix}$$

$$\det(A) = -12.$$

Definition 3.11: The Determinant of an $n \times n$ Matrix

Let A *be an* $n \times n$ *matrix where* $n \geq 2$ *and suppose the determinant of an* $(n-1) \times (n-1)$ *has been defined. Then*

$$\det(A) = \sum_{j=1}^{n} a_{ij}\text{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\text{cof}(A)_{ij}$$

The first formula consists of expanding the determinant along the i^{th} *row and the second expands the determinant along the* j^{th} *column.*

The Determinant of a Triangular Matrix

Theorem 3.13: Determinant of a Triangular Matrix

Let A *be an upper or lower triangular matrix. Then* $\det(A)$ *is obtained by taking the product of the entries on the main diagonal.*

Example 3.14: Determinant of a Triangular Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Find $\det(A)$.

$$\det(A) = 1 \times 2 \times 3 \times (-1) = -6.$$

Properties of Determinants I: Examples

Definition 3.15: Row Operations

The row operations consist of the following

1. *Switch two rows.*
2. *Multiply a row by a nonzero number.*
3. *Replace a row by a multiple of another row added to itself.*

Theorem 3.16: Switching Rows

Let A be an $n \times n$ matrix and let B be a matrix which results from switching two rows of A. Then $\det(B) = -\det(A)$.

Example 3.17: Switching Two Rows

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and let* $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. *Knowing that* $\det(A) = -2$, *find* $\det(B)$.

$\det(A) = 1 \times 4 - 3 \times 2 = -2.$
$\det(B) = -\det(A) = -(-2) = 2.$

Theorem 3.18: Multiplying a Row by a Scalar

Let A be an $n \times n$ matrix and let B be a matrix which results from multiplying some row of A by a scalar k. Then $\det(B) = k\det(A)$.

Theorem 3.19: Scalar Multiplication

Let A and B be $n \times n$ matrices and k a scalar, such that $B = kA$. Then $\det(B) = k^n \det(A)$.

Example 3.20: Multiplying a Row by 5

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}$. *Knowing that* $\det(A) = -2$, *find* $\det(B)$.

$\det(A) = -2$.
$\det(B) = 5 \times \det(A) = 5 \times -2 = -10$.

Theorem 3.21: Adding a Multiple of a Row to Another Row

Let A be an $n \times n$ matrix and let B be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.

Example 3.22: Adding a Row to Another Row

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and let* $B = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$. *Find* $\det(B)$.

Solution. By Definition 3.1, $\det(A) = -2$. Notice that the second row of B is two times the first row of A added to the second row. By Theorem 3.16, $\det(B) = \det(A) = -2$. As usual, you can verify this answer using Definition 3.1. ♠

Example 3.23: Multiple of a Row

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. *Show that* $\det(A) = 0$.

Solution. Using Definition 3.1, the determinant is given by

$$\det(A) = 1 \times 4 - 2 \times 2 = 0$$

Theorem 3.24: Determinant of a Product

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

Example 3.25: The Determinant of a Product

Compare $\det(AB)$ *and* $\det(A)\det(B)$ *for*

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}$$

$$\det(AB) = \det \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix} = -40$$

$$\det(A) = \det \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} = 8$$

$$\det(B) = \det \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = -5$$

$\det(A)\det(B) = 8 \times (-5) = -40 = \det(AB)$.

Theorem 3.26: Determinant of the Transpose

Let A be a matrix where A^T is the transpose of A. Then,

$$\det\left(A^T\right) = \det(A)$$

Example 3.27: Determinant of the Transpose

Let

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$$

Find $\det\left(A^T\right)$.

$$A^T = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$$

$\det\left(A^T\right) = 2 \times 3 - 5 \times 4 = -14$. Hence, $\det(A) = \det\left(A^T\right)$.

Theorem 3.28: Determinant of the Inverse

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$. If this is true, it follows that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Example 3.29: Determinant of an Invertible Matrix

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$. *For each matrix, determine if it is invertible. If so, find the determinant of the inverse.*

$\det(A) = 3 \times 4 - 2 \times 6 = 12 - 12 = 0$

A is not invertible.

$\det(B) = 2 \times 1 - 5 \times 3 = 2 - 15 = -13$
B is invertible and the determinant of the inverse is given by

$$\begin{aligned}\det\left(A^{-1}\right) &= \frac{1}{\det(A)} \\ &= \frac{1}{-13} \\ &= -\frac{1}{13}\end{aligned}$$

Properties of Determinants II: Some Important Proofs

Lemma 3.31:

If A is an $n \times n$ matrix such that one of its rows consists of zeros, then $\det A = 0$.

Theorem 3.33:

Let A and B be $n \times n$ matrices.

1. *If A is obtained by interchanging ith and jth rows of B (with $i \neq j$), then $\det A = -\det B$.*
2. *If A is obtained by multiplying ith row of B by k then $\det A = k \det B$.*
3. *If two rows of A are identical then $\det A = 0$.*
4. *If A is obtained by multiplying ith row of B by k and adding it to jth row of B $(i \neq j)$ then $\det A = \det B$.*

Theorem 3.34:

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

Theorem 3.35:

Let A be a matrix where A^T is the transpose of A. Then,

$$\det\left(A^T\right) = \det(A)$$

Finding Determinants using Row Operations

Example 3.37: Finding a Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 11 & 22 \\ 0 & -3 & -8 & -13 \\ 0 & 6 & 30 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 14 & -17 \end{bmatrix} \qquad \det(D) = 1(-3) \begin{vmatrix} 11 & 22 \\ 14 & -17 \end{vmatrix} = 1485$$

Example 3.38: Find the Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & -3 & 2 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & -4 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -1 & -1 \\ 0 & -3 & -4 & 1 \\ 0 & -10 & -8 & -4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -8 & 3 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

$$\det(D) = 1 \det \begin{bmatrix} 0 & -1 & -1 \\ -8 & -4 & 1 \\ 10 & -8 & -4 \end{bmatrix} + 0 + 0 + 0$$

Expanding again along the first column, we have

$$\det(D) = 1 \left(0 + 8 \det \begin{bmatrix} -1 & -1 \\ -8 & -4 \end{bmatrix} + 10 \det \begin{bmatrix} -1 & -1 \\ -4 & 1 \end{bmatrix} \right) = -82$$

Exercises

Exercise 3.1.2 *Let* $A = \left[\begin{array}{rrr} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{array}\right]$. *Find the following.*

(a) $minor(A)_{11}$

(b) $minor(A)_{21}$

(c) $minor(A)_{32}$

(d) $cof(A)_{11}$

(e) $cof(A)_{21}$

(f) $cof(A)_{32}$

Exercise 3.1.3 *Find the determinants of the following matrices.*

(a) $\left[\begin{array}{rrr} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{array}\right]$

(b) $\left[\begin{array}{rrr} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{array}\right]$

(c) $\left[\begin{array}{rrrr} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{array}\right]$

Exercise 3.1.4 *Find the following determinant by expanding along the first row and second column.*

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Exercise 3.1.5 *Find the following determinant by expanding along the first column and third row.*

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

Exercise 3.1.6 *Find the following determinant by expanding along the second row and first column.*

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Exercise 3.1.7 *Compute the determinant by cofactor expansion. Pick the easiest row or column to use.*

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

Exercise 3.1.8 *Find the determinant of the following matrices.*

(a) $A = \begin{bmatrix} 1 & -34 \\ 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 3 & 14 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 3 & 15 & 0 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 3.1.22 *Find the determinant using row operations to first simplify.*

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix}$$

Exercise 3.1.23 *Find the determinant using row operations to first simplify.*

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix}$$

Exercise 3.1.24 *Find the determinant using row operations to first simplify.*

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix}$$

Applications of the Determinant

A Formula for the Inverse

The determinant of a matrix also provides a way to find the inverse of a matrix. Recall the definition of the inverse of a matrix in Definition 2.33. We say that A^{-1}, an $n \times n$ matrix, is the inverse of A, also $n \times n$, if $AA^{-1} = I$ and $A^{-1}A = I$.

We now define a new matrix called the **cofactor matrix** of A. The cofactor matrix of A is the matrix whose ij^{th} entry is the ij^{th} cofactor of A. The formal definition is as follows.

Definition 3.39: The Cofactor Matrix

*Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. Then the **cofactor matrix of** A, denoted $cof(A)$, is defined by $cof(A) = \left[cof(A)_{ij}\right]$ where $cof(A)_{ij}$ is the ij^{th} cofactor of A.*

Theorem 3.40: The Inverse and the Determinant

Let A be an $n \times n$ matrix. Then

$$A\, adj(A) = adj(A)A = \det(A)I$$

Moreover A is invertible if and only if $\det(A) \neq 0$. In this case we have:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Example 3.41: Find Inverse Using the Determinant

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

using the formula in Theorem 3.40.

Solution. According to Theorem 3.40,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

By Theorem 3.21, $\det(A) = \det(B)$. By Theorem 3.13, $\det(B) = 1 \times -6 \times -2 = 12$. Hence, $\det(A) = 12$.

Now, we need to find $\operatorname{adj}(A)$. To do so, first we will find the cofactor matrix of A. This is given by

$$\operatorname{cof}(A) = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{12} \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Example 3.42: Find the Inverse From a Formula

Find the inverse of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$$

using the formula given in Theorem 3.40.

$$A^{-1} = \frac{1}{(1/6)} \operatorname{adj}(A) = 6 \operatorname{adj}(A)$$

$$A^{-1} = 6 \begin{bmatrix} \begin{vmatrix} \frac{1}{3} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} -\frac{1}{6} & -\frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} -\frac{1}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ -\begin{vmatrix} 0 & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} \end{bmatrix}^T$$

Expanding all the 2×2 determinants, this yields

$$A^{-1} = 6 \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Example 3.43: Inverse for Non-Constant Matrix

Suppose

$$A(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$$

Show that $A(t)^{-1}$ exists and then find it.

Solution. First note $\det(A(t)) = e^t(\cos^2 t + \sin^2 t) = e^t \neq 0$ so $A(t)^{-1}$ exists.

The cofactor matrix is

$$C(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}$$

and so the inverse is

$$\frac{1}{e^t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}^T = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

Exercise 3.2.1 *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

Exercise 3.2.2 *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

Exercise 3.2.3 *Let*

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

Exercise 3.2.4 *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

Exercise 3.2.5 *Let*

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

Exercise 3.2.14 *Find the inverse, if it exists, of the matrix*

$$A = \begin{bmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}$$

Exercise 3.2.7 *Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

Exercise 3.2.8 *Consider the matrix*

$$A = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.